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A Problem of Avoidance of a Rotating Segment

M. N. Ivanov and E. P. Maslov
Institute of Control Sciences, Moscow, Russia

Abstract—A planar problem of avoidance of a rotating segment is considered. Optimal avoidance trajectories and the boundary of the capture set are derived.

1. INTRODUCTION

We consider a planar problem of avoidance of a rotating segment. Two pointwise players, a searching system P and an evader E, are located in a plane. The player P is immobile and searches by way of rotation of a ("detection") segment, the end point of which is attached to P. The length of the segment is R; it rotates clockwise, with a constant angular velocity ω. The evader's goal is to avoid being captured by the detection segment.

The problem is solved under the following assumptions. The evader E is capable of instantaneous turns (*viz*, "simple motion" in Isaac's terminology, [1]); his velocity is restricted by an upper bound v. The player E has complete information about P; he knows the location of P on the plane, the length R of the "detection segment" and its initial orientation, its direction of rotation, and the value of its angular velocity ω. P is an automaton, i.e., P exerts no control.

The problem is solved from E's point of view and its complete solution is obtained. The following questions are answered analytically.

- Optimal laws of avoidance of the detection segment for arbitrary initial positions of E are derived.
- The delineation of the capture set—the domain of E's initial positions, from which avoidance is impossible—is performed.

In our opinion, the problem on hand is at most superficially similar to the "Lady in the Lake" differential game [1,2]. In that game, the pursuer P, moving with unit velocity, is confined to the perimeter of a circular lake of radius R. The evader E, which is inside the circle, moves with a velocity of $v < 1$; E's goal is to land on the perimeter as far as possible from P.

The problem of avoiding a rotating detection zone was also considered in [3,4].

2. PROBLEM STATEMENT

We define a polar frame of reference, with the origin located at P; the polar (y-) axis L coincides with the initial position of the detection segment (Figure 1). In what follows, angles are measured clockwise. The equations of E's motion are as follows:

$$\begin{aligned} \dot{\rho} &= u\cos\psi, \\ \dot{\alpha} &= \frac{u}{\rho}\sin\psi, \end{aligned} \tag{1}$$

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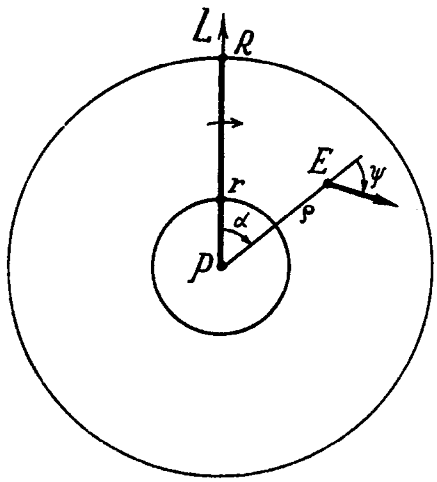

Figure 1.

where (α, ρ) denote the polar coordinates of the evader. In this system, u is a value of E's instantaneous linear velocity, subjected to the constraint $0 \leq u \leq v$; ψ is E's leading angle, measured relative to the line PE. The pair (u, ψ) forms the control vector of the evader.

At the instant $t = 0$, the evader E is located at the initial point

$$\alpha(0) = \alpha_0, \quad \rho(0) = \rho_0; \qquad \text{with} \tag{2}$$

$$\rho_0 < R. \tag{3}$$

Let the inequality (3) hold. Then player E avoids detection if either one of the following events takes place:

- the evader leaves the circular disc $S_R(\mathrm{P})$—centered at the pole P, with radius R—before the detection segment captures E;
- the evader remains inside the circular set $S_R(\mathrm{P})$ during an infinite time interval, ensuring avoidance from the detection segment; this possibility holds if, beginning from an instant $t = \tau$ (possibly, $\tau = 0$), E equates his angular rate with respect to the pole P with the angular velocity ω of the detection segment, and up to the instant $t = \tau$, E avoids the detection segment.

E's maximal linear velocity is equal to v. Being at a distance ρ from the pole P, the evader can guarantee an angular velocity of at most v/ρ. Thus, the problem of avoidance makes sense if the inequality

$$\rho_0 > \frac{v}{\omega} \triangleq r$$

holds. If the inequality does not hold, then E's rotation with respect to P (i.e., with $\psi(t) = \frac{\pi}{2}$) and a linear velocity $u = \omega\, \rho_0$ ensure the avoidance of detection.

Consequently, the problem of avoidance is meaningful only for those values v, ω, R, which satisfy the inequality $R > v/\omega$. For these values of parameters, the set of E's initial positions (α_0, ρ_0), meeting the inequalities

$$R > \rho_0 > \frac{v}{\omega}, \tag{4}$$

is not empty.

3. SOLUTION OF THE PROBLEM

The avoidance problem is decomposed into two sub-problems:

- a problem of optimal control, which ensures avoidance by fleeing to the exterior of $S_R(\mathrm{P})$; we denote this case by A;

- a problem of optimal control, ensuring avoidance inside the set $S_R(\mathrm{P})$; in this case, the evader heads to the circle radius $r = v/\omega$ and subsequently rotates with respect to P; this is case B.

Cases A and B are considered separately. The optimal avoidance trajectories are defined for each case. The sets $\mathbb{A}$ and $\mathbb{B}$ of E's initial positions, from which avoidance outside and inside the set $S_R(\mathrm{P})$, respectively, is impossible, are delineated. The intersection of the sets $\mathbb{A}$ and $\mathbb{B}$ constitutes the capture set—a domain of E's initial positions, from which avoidance of detection by a rotating segment is in general impossible; we denote this set by $\mathbb{C}$.

3.1. Avoidance to the Exterior of the Set $S_R(\mathrm{P})$

Let T be the time instant where E reaches the boundary of $S_R(\mathrm{P})$; it is defined by the relation

$$\rho(T) - R = 0. \tag{5}$$

We introduce a payoff

$$G(T) = \alpha(T) - \omega T, \tag{6}$$

which is the angle between the line PE and the detection segment at time T. Avoidance takes place if $G(T) > 0$; we recall that the detection segment rotates clockwise.

Thus, the problem is reduced to the optimal control problem with the dynamics (1), initial condition (2), terminal condition (5), and payoff (6):

$$G(T) \to \max_{u,\psi}.$$

ASSERTION 1. *The optimal trajectory of E is a straight line, tangent to the circle of radius r (Figure 2). The evader moves along this line with the maximal velocity $u^* = v$; here, and in what follows, the asterisk denotes an optimal value. Hence, in the polar frame, the optimal control ψ^* is*

$$\psi^*(t) = \arctan \frac{v/\omega}{\sqrt{\rho_0^2 - (v/\omega)^2} + v\,t}. \tag{7}$$

The duration of the avoidance maneuver is

$$T^* = \frac{1}{v}\left[\sqrt{R^2 - \left(\frac{v}{\omega}\right)^2} - \sqrt{\rho_0^2 - \left(\frac{v}{\omega}\right)^2}\right]. \tag{8}$$

The optimal criterion value is

$$G^*(T^*) = \alpha_0 - \arccos\left[\frac{v}{\omega\,\rho_0}\right] + \arccos\left[\frac{v}{\omega\,R}\right] - \sqrt{\left(\frac{\omega\,R}{v}\right)^2 - 1} + \sqrt{\left(\frac{\omega\,\rho_0}{v}\right)^2 - 1}. \tag{9}$$

The proof of Assertion 1 is given in the Appendix.

Let us now construct a curve $\mathcal{L}_1$—a boundary of the set $\mathbb{A}$ of E's initial positions, from which avoidance to the exterior of $S_R(\mathrm{P})$ is impossible. The points of the curve satisfy the condition $G^*(T^*) = 0$; thus the equation of the curve is as follows:

$$\alpha_0 - \arccos\left[\frac{v}{\omega\,\rho_0}\right] + \arccos\left[\frac{v}{\omega\,R}\right] - \sqrt{\left(\frac{\omega\,R}{v}\right)^2 - 1} + \sqrt{\left(\frac{\omega\,\rho_0}{v}\right)^2 - 1} = 0. \tag{10}$$

The polar equation (10) defines an evolvent [5] of the r-circle. The evolvent passes through the points with coordinates ($\alpha_0 = 0$, $\rho_0 = R$) and ($\hat{\alpha}_0 = \tan\gamma - \gamma$, $\rho_0 = v/\omega \triangleq r$), where $\gamma = \arccos[v/(\omega\,R)]$. Avoidance of detection on the trajectories leading to the exterior of $S_R(\mathrm{P})$ is impossible if the evader's initial positions belong to the interior of the set $\mathbb{A}$, bounded by the following lines (Figure 2):

- the curve $\mathcal{L}_1$;
- the polar axis L;
- an arc of the r-circle, centered at the point P.

For such initial positions $G^*(T^*) < 0$.

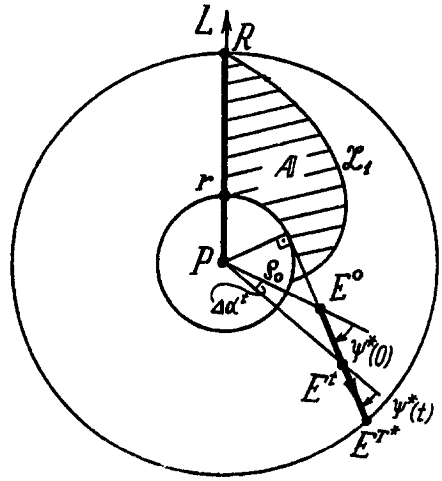

Figure 2.

3.2. Avoidance Inside the Set S_R(P)

Avoidance inside the set S_R(P) entails the evader's passage to the boundary of the r-circle, centered at the pole P, with subsequent rotation (with linear velocity v) about P. Thus, we have a problem of optimal control with the equations of motion (1), initial condition (2), payoff (6), and terminal condition

$$\rho(\tau) - \frac{v}{\omega} = 0. \tag{11}$$

ASSERTION 2. *The solution of the optimization problem (1), (2), (6), (11) is as follows. The optimal trajectory of E is a straight line, tangent to the r-circle. The evader moves along this line with the maximal velocity $u^* = v$, and the maneuver's duration is*

$$\tau^* = \frac{1}{v}\sqrt{\rho_0^2 - \left(\frac{v}{\omega}\right)^2}. \tag{12}$$

The optimal control ψ^* has the following form:

$$\psi^*(t) = \arctan\frac{v/\omega}{vt - \sqrt{\rho_0^2 - \left(\frac{v}{\omega}\right)^2}}, \qquad t \in [0, \tau^*]. \tag{13}$$

The optimal value of the criterion is

$$G^*(\tau^*) = \alpha_0 + \arccos\left[\frac{v}{\omega\,\rho_0}\right] - \sqrt{\left(\frac{\omega\,\rho_0}{v}\right)^2 - 1}. \tag{14}$$

The proof of Assertion 2 is similar to the proof of Assertion 1, and is omitted.

If E's initial position is such that $G^*(\tau^*) > 0$, then avoidance takes place. In that case, beginning from $t = \tau^*$, the evader rotates with linear velocity v along the r-circle, centered at the pole P; the optimal control is

$$\psi^*(t) = \frac{\pi}{2}, \qquad t \geq \tau^*. \tag{15}$$

We now embark on the construction of a curve $\mathcal{L}_2$—a boundary of the set $\mathbb{B}$ of E's initial positions, from which avoidance inside S_R(P) is impossible. Points on this curve satisfy the condition $G^*(\tau^*) = 0$; thus the equation of the curve is as follows:

$$\alpha_0 + \arccos\left[\frac{v}{\omega\,\rho_0}\right] - \sqrt{\left(\frac{\omega\,\rho_0}{v}\right)^2 - 1} = 0. \tag{16}$$

The polar equation (16) defines an evolvent of the r-circle, which passes through the points ($\alpha_0 = 0$, $\rho_0 = v/\omega \triangleq r$) and ($\hat{\alpha}_0 = \tan\gamma - \gamma$, $\rho_0 = R$). Avoidance of detection on the trajectories leading inside the set S_R(P) is impossible if the evader's initial positions belong to the interior of a set $\mathbb{B}$, which is bounded by the following lines (Figure 3):

- the curve $\mathcal{L}_2$;
- the polar axis L;
- an arc of the R-circle, centered at the pole P.

For such initial positions $G^*(\tau^*) < 0$.

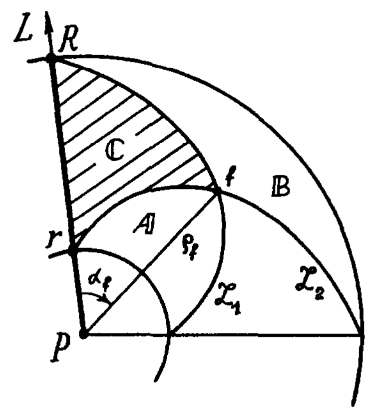

Figure 3.

The intersection of the sets $\mathbb{A}$ and $\mathbb{B}$ constitutes the capture set $\mathbb{C}$. From the inner points of the capture set $\mathbb{C}$, avoidance of detection is impossible—neither to the exterior of the set S_R(P), nor inside this set. The set $\mathbb{C}$ is bounded by the polar axis L and the curves $\mathcal{L}_1$, $\mathcal{L}_2$ (Figure 3). The point of intersection of the curves $\mathcal{L}_1$ and $\mathcal{L}_2$ (we denote it by f) has the coordinates (α_f, ρ_f) which are derived as a solution of the system of equations:

$$\begin{aligned} 2\alpha_f + \arccos\left[\frac{v}{\omega R}\right] - \sqrt{\left(\frac{\omega R}{v}\right)^2 - 1} = 0, \\ \alpha_f + \arccos\left[\frac{v}{\omega \rho_f}\right] - \sqrt{\left(\frac{\omega \rho_f}{v}\right)^2 - 1} = 0. \end{aligned} \tag{17}$$

If $\alpha_f \le 2\pi$, then f is a corner point of the set $\mathbb{C}$. Otherwise, the point f is absent and the curves $\mathcal{L}_1$, $\mathcal{L}_2$ reach the polar axis L. It follows from the first equation of the system (17) that $\alpha_f \le 2\pi$ if $v/(\omega R) \ge 0.07091346\dots$.

4. A SUBOPTIMAL AVOIDANCE STRATEGY

Consider the case where the evader steers to the exterior of the circular set S_R(P), and let T^* be the time instant where E reaches the R-circle—*viz.*, the boundary of the set S_R(P). It follows from Assertion 1 and equations (7) and (8), that the optimal control ψ^* satisfies the condition

$$\sin\psi^*(T^*) = \frac{v}{\omega R} \triangleq \frac{r}{R}. \tag{18}$$

As $r/R \to 0 : \psi^*(T^*) \to 0$. Thus, for small values of the quotient r/R, the optimal E trajectory leading outside the set S_R(P), is practically directed along a radius of the R-circle (Figure 4).

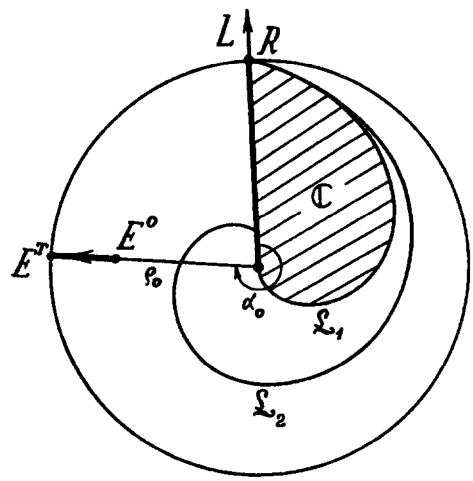

Figure 4.

In what follows, the E's avoidance strategy, meeting the condition $\psi^*(t) \equiv 0$ is referred to as suboptimal. Namely, this strategy was discussed in [6].

As far as the strategy of avoidance is concerned, we devote our attention to the delineation of the capture set $\mathbb{C}$.

Trajectories of E's motion are defined, as before, by the system (1). But, in contrast to inequalities (4), E's initial position (ρ_0, α_0) in the problem at hand meets the condition

$$0 < \rho_0 < R. \tag{19}$$

The evader's motion along the straight line, which coincides its initial position and the pole P, corresponds to the control $\psi = 0$ (Figure 4). Integrating the system (1) with $\psi = 0$, we derive

$$\begin{aligned} \rho(t) &= \rho_0 + v\,t, \\ \alpha(t) &= \alpha_0. \end{aligned} \tag{20}$$

In accordance with relations (6) and (20), at the instant T defined by the relation $\rho\,(T) = R$, the value of the criterion is

$$G(T) = \alpha(T) - \omega\,T = \alpha_0 - \frac{\omega}{v}\,(R - \rho_0). \tag{21}$$

The points of the capture set boundary satisfy the condition $G(T) = 0$. Thus, the equation of a line $\mathcal{L}$, which delimits the capture set, has the form:

$$\rho_0 = R - \frac{v\,\alpha_0}{\omega}. \tag{22}$$

Equation (22) defines a spiral of Archimedes [5]. As long as the initial position of the evader satisfies equation (19), the curve $\mathcal{L}$ always passes through the point $(\alpha_0 = 0,\ \rho_0 = R)$. The value $\rho(\alpha_0) = 0$ corresponds to the angle $\alpha_0 = \omega\,R/v$. But the E's initial angular coordinate satisfies the condition $0 < \alpha_0 < 2\pi$. Thus, if the inequality

$$\frac{\omega\,R}{v} \le 2\pi \tag{23}$$

holds, then the curve $\mathcal{L}$ (denoted by $\mathcal{L}_1$ in Figure 4) is completed at the point P. Otherwise, the usable part of the curve $\mathcal{L}$ (denoted by $\mathcal{L}_2$) is completed at the point of the polar axis with the coordinate $\hat{\rho} = R - 2\pi\,v/\omega$.

Let $\omega R/v \leq 2\pi$; then the area of the capture set $\mathbb{C}$ is equal to

$$S = \frac{\omega R^3}{6v}.$$

Otherwise, the area of the capture set is equal to

$$S = \frac{\left[(2\pi\, v - R\,\omega)^3 + (R\,\omega)^3\right]}{6\,\omega^2\, v}.$$

The areas quotient $S/(\pi\, R^2)$ is an estimate of the quality of the avoidance strategy.

APPENDIX
PROOF OF ASSERTION 1

In accordance with the maximum principle [7], the solution of the maximization problem (1), (2), (5), and (6) is as follows. We form the Hamiltonian:

$$H = \lambda_1\, u\, \cos\psi + \frac{\lambda_2}{\rho}\, u \sin\psi \to \min_{u,\psi}. \tag{A.1}$$

The vector (λ_1, λ_2) of costate variables satisfies the adjoint differential system

$$\begin{aligned} \dot{\lambda}_1 &= \frac{\lambda_2}{\rho^2}\, u \sin\psi, \\ \dot{\lambda}_2 &= 0. \end{aligned} \tag{A.2}$$

The transversality condition at the instant $t = T^*$ is as follows:

$$\delta G + \lambda_1\, \delta\rho + \lambda_2\, \delta\alpha - H^*\, \delta t = 0, \tag{A.3}$$

where $\delta G = \delta\alpha - \omega\delta t$ and the variation $\delta\rho$ satisfies the condition $\delta\rho = 0$. It follows from these relations that

$$\lambda_2(t) = \lambda_2(T^*) = -1, \tag{A.4}$$
$$H^*(T^*) = -\omega. \tag{A.5}$$

By virtue of the formula (A.5), $\lambda_1^2 + (\lambda_2/\rho)^2 \neq 0$ at any $0 \leq t \leq T^*$; thus, we have for the optimal control vector:

$$u^* = v, \tag{A.6}$$

$$\sin\psi^* = -\frac{\lambda_2/\rho}{\sqrt{\lambda_1^2 + (\lambda_2/\rho)^2}}, \quad \text{and} \tag{A.7}$$

$$H^* = -v\sqrt{\lambda_1^2 + \left(\frac{\lambda_2}{\rho}\right)^2}. \tag{A.8}$$

It follows from these relations, that the evader moves with the maximal velocity, and at the instant $t = T^*$ the condition

$$\sin\psi^*(T^*) = \frac{v}{\omega\, R} \triangleq \frac{r}{R} \geq 0 \tag{A.9}$$

holds.

Rewriting the system (1) in a Cartesian frame and using the maximum principle, we find that the optimal trajectory of E's motion is a straight line, along which it moves with a maximal veloctiy $u^* = v$.

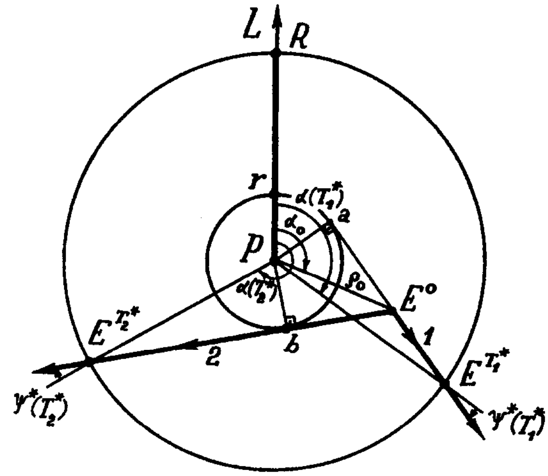

Figure 5.

At the instant $t = T^*$, as E reaches the R-circle, the condition (A.7) is fulfilled. It means that E's optimal trajectory is a tangent to the circle centered at the point P with radius $r = v/\omega$. But from each initial position (α_0, ρ_0) of E, two straight lines, which are tangents to the r-circle, can be drawn. Further, on each tangent, an avoidance trajectory satisfying the necessary optimality condition $\psi^*(T^*) \geq 0$ can be constructed. The first trajectory is such that its extension touches the r-circle; the second one touches the r-circle. The trajectories are shown in Figure 5, and they are marked by symbols 1 and 2, respectively. It is necessary to carry out an additional analysis with a view to determine the trajectory that maximizes the functional $G(T)$.

We consider the trajectories separately. Let T_1^* and T_2^* be the times of E's motion along Trajectories 1 and 2, respectively, and $E^{T_1^*}$ and $E^{T_2^*}$ the corresponding positions of the evader at the instants of reaching the R-circle.

TRAJECTORY 1. We find the time T_1^* of E's motion from the triangles $\triangle\, \mathrm{P}a\mathrm{E}^0$ and $\triangle\, \mathrm{P}a\mathrm{E}^{T_1^*}$ (Figure 5):

$$T_1^* = \frac{1}{v}\left[\sqrt{R^2 - \left(\frac{v}{\omega}\right)^2} - \sqrt{\rho_0^2 - \left(\frac{v}{\omega}\right)^2}\right]. \tag{A.10}$$

It follows from the same triangles that the angle $\alpha(T_1^*)$, defined by the L-axis and the straight line meeting at the points P and $\mathrm{E}^{T_1^*}$ is equal to

$$\alpha^*(T_1^*) = \alpha_0 - \arccos\left[\frac{v}{\omega\,\rho_0}\right] + \arccos\left[\frac{v}{\omega\,R}\right]. \tag{A.11}$$

Thus we have

$$G^*(T_1^*) = \alpha_0 - \arccos\left[\frac{v}{\omega\,\rho_0}\right] + \arccos\left[\frac{v}{\omega\,R}\right] - \sqrt{\left(\frac{\omega\,R}{v}\right)^2 - 1} + \sqrt{\left(\frac{\omega\,\rho_0}{v}\right)^2 - 1}. \tag{A.12}$$

TRAJECTORY 2. We find from the triangles $\triangle\, \mathrm{PE}^0 b$ and $\triangle\, \mathrm{P}b\mathrm{E}^{T_2^*}$:

$$T_2^* = \frac{1}{v}\left[\sqrt{R^2 - \left(\frac{v}{\omega}\right)^2} + \sqrt{\rho_0^2 - \left(\frac{v}{\omega}\right)^2}\right], \tag{A.13}$$

$$\alpha^*(T_2^*) = \alpha_0 + \arccos\left[\frac{v}{\omega\,\rho_0}\right] + \arccos\left[\frac{v}{\omega\,R}\right]. \tag{A.14}$$

Thus,

$$G^*(T_2^*) = \alpha_0 + \arccos\left[\frac{v}{\omega\,\rho_0}\right] + \arccos\left[\frac{v}{\omega\,R}\right] - \sqrt{\left(\frac{\omega\,R}{v}\right)^2 - 1} - \sqrt{\left(\frac{\omega\,\rho_0}{v}\right)^2 - 1}. \tag{A.15}$$

Consequently, the evader chooses Trajectory 1, if $G(T_1^*) > G(T_2^*)$, and Trajectory 2, otherwise.

It follows from (A.12) and (A.15) that $G(T_1^*) > G(T_2^*)$, if

$$\sqrt{\left(\frac{\omega\,\rho_0}{v}\right)^2 - 1} - \arccos\left[\frac{v}{\omega\,\rho_0}\right] > 0. \tag{A.16}$$

Thus, the evader uses Trajectory 1 for $\rho_0 \geq \bar{\rho}_0$, where $\bar{\rho}_0$ is a root of the equation

$$\sqrt{\left(\frac{\omega\,\rho_0}{v}\right)^2 - 1} - \arccos\left[\frac{v}{\omega\rho_0}\right] = 0. \tag{A.17}$$

But this equation has in the range $[v/\omega; R]$ a unique solution, $\bar{\rho}_0 = v/\omega$. Indeed, with regard to the notation

$$\zeta = \arccos\left[\frac{v}{\omega\rho_o}\right], \qquad 0 \leq \zeta \leq \arccos\left[\frac{v}{\omega R}\right] < \frac{\pi}{2},$$

equation (A.17) can be represented in the following form:

$$\tan\zeta - \zeta = 0. \tag{A.18}$$

And this equation in the range $\zeta \in [0; \frac{\pi}{2}]$ has the unique root $\zeta = 0$; so $\bar{\rho}_0 = v/\omega$. It means that in the problem on hand, the player E always selects Trajectory 1 because it ensures the maximal value of $G(T)$: for any $\rho_0 \in [v/\omega; R]$, $G(T_1^*) > G(T_2^*)$, and so $G(T^*) = G(T_1^*)$.

Now, we derive a relation defining the optimal control $\psi^*(t)$ as a function of the time t. We recall that $\psi(t)$ is the angle included between the line PE and E's velocity vector.

Let E^t denote the position of the evader at an arbitrary instant t. It follows from the triangle $\triangle\,\mathrm{PE}^0\mathrm{E}^t$ (Figure 2) that

$$\rho_0\,\sin(\Delta\alpha^t) = v\,t\,\sin\psi^*(t),$$

where

$$\Delta\alpha^t = \psi^*(0) - \psi^*(t),$$
$$\sin\psi^*(0) = \frac{v}{\omega\,\rho_0}, \qquad \cos\psi^*(0) > 0.$$

Thus,

$$\psi^*(t) = \arctan\frac{v/\omega}{\sqrt{\rho_0^2 - (v/\omega)^2} + v\,t}, \qquad t \in [0; T^*],$$

where the value of T^* is given by formula (A.10).

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